ON CHANGES IN ZONAL MOMENTUM IN SHORT-RANGE NUMERICAL PREDICTION

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[Manuscript received December 29, 1959; revised February 19, 1960]

ABSTRACT

Different factors influencing the changes in the zonally averaged wind are investigated. It is evident from recent investigations of errors in the zonally averaged winds in the non-divergent, one-parameter model that the convergence of the meridional transport of zonal momentum concentrates too much momentum in the middle latitudes and predicts too small amounts of momentum in the low and high latitudes.

The other factors influencing the changes of zonal momentum, i.e., mean meridional circulations, friction, and vertical transport of momentum, are investigated on an averaged basis in the following sections: In section 3 it is shown that the divergent, one-parameter model will reduce the errors in zonal momentum predicted by the non-divergent model, due to an implied mean meridional circulation. The corrections to the predictions of changes in zonal momentum caused by mean meridional circulations in a two-parameter model are investigated in section 4 by the aid of operationally computed initial values of the vertical velocities. It is shown that reductions in the errors of the non-divergent model with respect to zonal momentum can be expected with a careful arrangement of the information levels in a two-parameter model.

Section 5 contains a similar investigation of the averaged contribution of vertical advection to changes in zonal momentum. It is found that this contribution is smaller than the one resulting from mean meridional circulations, and further that the contribution from the vertical advection of momentum is not likely to reduce the errors found in the non-divergent predictions.

The main conclusion from the study is that the contributions from mean meridional circulations and surface friction are the most important for the reductions of errors in the prediction of zonal momentum in the non-divergent model. Some reduction of the errors can be expected in the divergent, one-parameter model or in a two-parameter model with a proper arrangement of the information levels. In order to incorporate surface friction in a realistic way, and further in order to avoid the artificial constraint of a non-divergent level appearing in a two-parameter model, it is most likely that more than two parameters are needed for accurate forecasts of zonal momentum.

1. INTRODUCTION

The expansion of the area used in operational numerical predictions has shown a number of problems connected with forecasts of the very large-scale features of the flow in the atmosphere. The problems connected with forecasts of the motion of the planetary waves have been treated in earlier investigations by Wolff [13], Cressman [3], and the author [11]. Another problem is connected with the prediction of the zonally-averaged flow; i.e., the mean zonal wind.

The errors in the zonal winds as predicted by the non-divergent barotropic model have recently been investigated in a very instructive manner by Bristor [1] who shows that the error pattern taken daily or in the average over a month has a very systematic distribution with latitude. By and large, the errors may be characterized by saying that the non-divergent barotropic model predicts too weak averaged zonal winds in the very low and the very high latitudes and too strong mean zonal winds in the middle latitudes. The error pattern is persistent month after month in the period analyzed by Bristor [1] and he suggests that the cause of these errors is the lack of momentum sources and sinks in the barotropic model.

If the 500-mb. flow is considered as a representation of the vertically averaged flow of the atmosphere, it follows, as shown for instance in [1], that surface friction is the only source of momentum which can influence the rate of change of the zonal wind apart from the effect of the convergence of the meridional transport of zonal momentum. The latter effect is, however, incorporated in the barotropic model. In this formulation we could therefore ascribe the errors in the predicted 500-mb. mean zonal winds to the neglect of the effects caused by surface friction.

On the other hand, if we consider the 500-mb. flow as a divergent flow, we may get changes in the mean zonal winds caused by the effects of the Coriolis term in the equations of motion, or in other words by the effects of a mean meridional circulation. In experiments on the general circulation of the atmosphere, Phillips [8] showed that the effects of the mean meridional circulation very nearly balance the effects of surface friction at the lower level (750 mb.), while the mean meridional circulation at the upper level acts opposite to the effects of the meridional transport of momentum at the upper level (250 mb.). The effects of the mean meridional circulation in a two-parameter model are therefore to decrease the mean zonal wind in the middle-latitudes at the upper level and to increase the wind at the lower level.

In view of these results it seems that we have two equally important factors in the lower troposphere and one in the higher troposphere to change the forecasts, which would be produced considering only the horizontal part of the motion. It also appears that these two factors, friction and mean meridional circulation, must be considered together because they act in opposite directions in the lower part of the atmosphere. If we use only two parameters to represent the vertical structure of the atmosphere, there will always be a level in the model where the effects of the mean meridional circulation vanish, the so-called non-divergent level. It seems very important to place this level as close as possible to the actual position of the corresponding level in the atmosphere, because otherwise we will ascribe errors, which appear in the model, solely to surface friction effects, although they actually could be caused by vertical velocities not included in the model.

It is obvious that great errors in the prediction of mean zonal winds can have a marked influence on the disturbances and their motion. From data in [1] one sees that an error of about 10 knots in 72 hours appears even on the monthly average for January and February 1958. If we for a moment assume that disturbances move approximately with the wind speed and that a mean error of 5 knots exists during a 72-hour period, it means that troughs and ridges are displaced about 650 km. too much to the east in the middle latitudes and a similar amount too far toward the west in the low latitudes. The result is that the troughs and ridges get a greater positive tilt (SW-NE tilt), which means that still more momentum is transported into the middle latitudes causing a further error in the tilt. We are therefore here dealing with a type of systematic, accumulating error, which, if strong enough, will result in a greater and greater tilt of the systems.

A special aspect of such a systematic error comes up in connection with the objective analysis scheme [4]. The 12-hour barotropic forecasts are used as a first guess for a subsequent 500-mb. analysis. It may happen that a sufficient amount of data does not exist to remove the systematic error in the first guess completely, with the result that even the initial analysis for the next forecast contains too strong zonal winds in the middle latitudes and too weak zonal winds to the north and south of the middle latitudes. This effect has actually been noticed in a comparison made between analyses produced by the present objective analysis scheme and analyses prepared by conventional methods.

It is the purpose of the following sections to investigate the possibilities which we have to remove these systematic errors from the barotropic forecasts. We have two effects, which should be considered: The effects of mean meridional circulations and surface friction. In section 3 we show that the present divergent, one-parameter model tends to reduce the errors in the mean zonal wind obtained from a non-divergent, one-parameter model. Finally, we present an analysis of the vertical motions computed from a two-parameter model, which shows that we can expect an improvement in the prediction of zonally averaged winds with a proper choice of the non-divergent level.

2. GENERAL AND SIMPLIFIED PROGNOSTIC EQUATIONS FOR THE MEAN ZONAL FLOW

The first equation of motion may be written in the following momentum form in pressure coordinates:

$$\frac{\partial u}{\partial t}+\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial u\omega}{\partial p}=-\frac{\partial \phi}{\partial x}+fv. \tag{2.1}$$

u, v, and ω are here the three components of the velocity vector; $\phi=gz$ the geopotential; f the Coriolis parameter; g the acceleration of gravity. No assumptions are made regarding the velocity components which may contain divergent as well as non-divergent components.

We define the following averaging operator:

$$\overline{(\ )}=\frac{1}{L}\int_0^L (\ )\,dx \tag{2.2}$$

where L is the length of the latitude circle.

Applying (2.2) to (2.1) we obtain

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial \overline{uv}}{\partial y}+\frac{\partial \overline{u\omega}}{\partial p}=+f\bar{v}. \tag{2.3}$$

It should be noted that an equation quite similar to (2.3) can be obtained by averaging the vorticity equation in its complete form:

$$\frac{\partial \zeta}{\partial t}+\nabla\cdot(\eta\mathbf{V})+\frac{\partial}{\partial x}\left(\omega\frac{\partial v}{\partial p}\right)-\frac{\partial}{\partial y}\left(\omega\frac{\partial u}{\partial p}\right)=0. \tag{2.4}$$

We now obtain applying (2.2):

$$\frac{\partial}{\partial y}\left[\frac{\partial \bar{u}}{\partial t}+\frac{\partial \overline{uv}}{\partial y}+\frac{\partial \overline{u\omega}}{\partial p}-f\bar{v}\right]=0. \tag{2.5}$$

The last term on the left side of (2.3), the effect of vertical transport of momentum, is seldom incorporated in short-range prediction. We later present evidence that justifies the neglect of this term at least in two-parameter models. The neglect of the term corresponds to disregarding the vertical advection and the twisting term in the vorticity equation. If the flow further is assumed to be strictly non-divergent and frictionless, we get the following simplified form of the momentum equation, which applies in a non-divergent, one-parameter model:

$$\frac{\partial}{\partial y}\left[\frac{\partial \bar{u}}{\partial t}+\frac{\partial \overline{uv}}{\partial y}\right]=0. \tag{2.6}$$

The term $\partial \overline{uv}/\partial y$ may of course in this case equally well be written $\overline{v\partial u/\partial y}$, because the second part

$$\overline{u\partial v/\partial y}=-\overline{\partial\left(\frac{1}{2}u^2\right)/\partial x}$$

vanishes identically.

A quasi-non-divergent model would apply a consistent form of the vorticity equation in the form (Wiin-Nielsen [10]):

$$\frac{\partial \zeta}{\partial t}+\mathbf{V}_\psi\cdot\nabla(\zeta+f)=-f_0\nabla\cdot\mathbf{V} \tag{2.7}$$

which would result in an equation for the rate of change of mean zonal wind of the form

$$\frac{\partial}{\partial y}\left(\frac{\partial \bar{u}}{\partial t}+\frac{\partial \overline{u_\psi v_\psi}}{\partial y}-f_0\bar{v}\right)=0 \tag{2.8}$$

which, compared with (2.5), would result in a correct form.

Using an inconsistent form of the vorticity equation

$$\frac{\partial \zeta}{\partial t}+\mathbf{V}_\psi\cdot\nabla(\zeta+f)=-\eta\nabla\cdot\mathbf{V} \tag{2.9}$$

where $\eta=\zeta+f$, f variable, would result in a zonal momentum equation as follows:

$$\frac{\partial}{\partial y}\left[\frac{\partial \bar{u}}{\partial t}+\frac{\partial \overline{u_\psi v_\psi}}{\partial y}-f\bar{v}\right]+\beta\bar{v}-\overline{\zeta\nabla\cdot\mathbf{V}}=0 \tag{2.10}$$

It is obvious from (2.10) that we may obtain fictitious changes in the zonal winds if the relative vorticity and the divergence are correlated along the latitude circles.

3. COMPARISON BETWEEN NON-DIVERGENT AND DIVERGENT ONE-PARAMETER MODELS

It has recently been found necessary to modify the non-divergent barotropic model in such a way that the motion of the ultra-long waves is greatly reduced. The modification is made by an estimate of the divergence term in the vorticity equation. The magnitude of the coefficient may be estimated in different ways. One way has been demonstrated by the author [11]. The argument for an introduction of a certain divergence in the one-parameter model has been the modification of the motion of the long waves. It is, however, obvious that we also by so doing change the prediction of the zonally-averaged winds, because we indirectly introduce a mean meridional circulation, which in turn has a certain effect on the zonally-averaged flow.

Considering the remarkable success of the divergent, one-parameter model in prediction of the 500-mb. flow pattern one gets the impression that this model in a fortunate way contains the essential features of the mid-tropospheric flow, at least with respect to the zonally-averaged changes predicted by the model. It is certainly true that the model does not have any mechanism which can predict rapid deepening or filling, because it does not include the effects of temperature advection, diabatic heating, and friction. Nevertheless, it will predict zonally-averaged changes, the results comparing favorably with predictions computed by models that contain these effects. The reason is of course that the effects, which have been disregarded, by and large compensate each other. We shall demonstrate this at the end of section 3.

The first purpose is to demonstrate that the long-wave modification also decreases the error in the zonal flow. Naturally, we could demonstrate this by a computation of forecasts with and without the long-wave modification, but in order to get a clear picture of the mechanism at work we shall consider a theoretical example.

We may picture the process which creates the error in the non-divergent forecast as follows. As a result of the positive tilt in the low latitudes we transport (too much) momentum into the middle latitudes. The negative tilt in the high latitudes in a similar way transports momentum into the middle latitudes. As this process, according to Bristor's [1] analysis of the errors, gives too great changes in the mean zonal winds, there must be some processes which counteract this effect.

In order to look into the effect of the divergence term in the present operational one-parameter model we shall consider an initial flow which has the characteristic tilts. Such a flow may be described by a perturbation stream function having the following form:

$$\psi(x,y)=A\cos\mu y\cos\kappa(x+\alpha y^2). \tag{3.1}$$

To simplify our computations we consider a rectangular region bounded to the north and south by walls and having an extension in the x-direction of one wavelength. The boundary conditions at the walls will be those given by Phillips [7]; i.e.

$$v=\frac{\partial \psi}{\partial x}=0 \text{ and } \frac{\partial}{\partial y}\left(\overline{\frac{\partial \psi}{\partial t}}\right)=0 \text{ at the walls.} \tag{3.2}$$

The arrangement of the coordinate system will be with the x-axis parallel to the walls and with the y-coordinate at the walls to be $y=\pm W$. With these definitions we have

$$\mu=\frac{\pi}{2W},\ \kappa=\frac{2\pi}{L} \tag{3.3}$$

where L is the wavelength in the x-direction. Now, the parameter α is related to the slope of the trough and ridge lines in the flow, these lines being defined by the condition

$v=0$. From the expression for ψ it is seen that the trough and ridge lines are curves described by the equation

$$x+\alpha y^2=0 \tag{3.4}$$

The curves are parabolas which for $\alpha>0$ are open toward the "west"; i.e., with a positive tilt in the lower half and a negative tilt in the upper half. It is also seen that the smaller α is, the smaller will be the tilt. The factor $\cos \mu y$ means that we assume a maximum disturbance amplitude in the middle of the channel and further that the boundary condition $v=0$, $y=\pm W$ is automatically satisfied.

The problem is now to predict the tendency in the mean zonal wind speed for the initial flow pattern. The vorticity equation applying to the divergent, one-parameter model has been discussed at length by the author [11]. The final form of the prognostic equation may be written:

$$\frac{\partial \zeta}{\partial t}-r^2\frac{\partial \psi}{\partial t}=J(\nabla^2\psi+f,\psi) \tag{3.5}$$

which averaged along the latitudes reduces to

$$\frac{d^2}{dy^2}\left(\frac{\partial \overline{\psi}}{\partial t}\right)-r^2\left(\frac{\partial \overline{\psi}}{\partial t}\right)=\frac{\partial^2(\overline{uv})}{\partial y^2}=\frac{\partial^2(\overline{u'v'})}{\partial y^2} \tag{3.6}$$

where u' and v' are the perturbation velocities. It is easily seen that

$$\overline{uv}=\overline{u'v'}, \text{ because } \overline{v}=\overline{\frac{\partial \psi}{\partial x}}\equiv 0$$

With the expression (3.1) for the perturbation stream function (3.6) becomes

$$\frac{d^2}{dy^2}\left(\frac{\partial \overline{\psi}}{\partial t}\right)-r^2\left(\frac{\partial \overline{\psi}}{\partial t}\right)=D\sin(2\mu y)+Ey\cos(2\mu y) \tag{3.7}$$

where

$$\begin{cases} D=2\alpha\mu\kappa^2A^2 \\ E=2\alpha\mu^2\kappa^2A^2 \end{cases} \tag{3.8}$$

The solution to (3.7) may be written in the form

$$\frac{\partial \overline{\psi}}{\partial t}=B_1\sin(2\mu y)+B_2y\cos(2\mu y)+C_1e^{ry}+C_2e^{-ry} \tag{3.9}$$

where

$$\begin{cases} B_1=-\dfrac{2\alpha}{\mu}\cdot\dfrac{\kappa^2A^2}{4+r^2/\mu^2}+\dfrac{8\alpha}{\mu}\cdot\dfrac{\kappa^2A^2}{(4+r^2/\mu^2)^2} \\ B_2=-2\alpha\cdot\dfrac{\kappa^2A^2}{4+r^2/\mu^2}\cdot \end{cases} \tag{3.10}$$

and where C_1 and C_2 are arbitrary constants to be determined from the boundary conditions

$$\frac{d}{dy}\left(\frac{\partial \overline{\psi}}{\partial t}\right)=0, \quad y=\pm W. \tag{3.11}$$

Differentiating (3.9) with respect to y and using (3.11) we arrive at a determination of C_1 and C_2 resulting in the values:

$$C_1=-C_2=\frac{1}{r}\cdot\frac{2\mu B_1+B_2}{2\cosh(rW)}. \tag{3.12}$$

Using (3.12) we can finally write the complete solution

$$\frac{\partial \overline{\psi}}{\partial t}=B_1\sin(2\mu y)+B_2y\cos(2\mu y)+\frac{2\mu B_1+B_2}{r}\cdot\frac{\sinh(ry)}{\cosh(rW)} \tag{3.13}$$

The main problem which we have in this section is to illustrate the difference between the non-divergent and divergent barotropic models. We shall therefore first consider the tendency in the mean zonal wind for the non-divergent model. This tendency can be computed from the expression

$$\frac{\partial \overline{u}}{\partial t}=-\frac{\partial \overline{u'v'}}{\partial y} \tag{3.14}$$

Corresponding to (2.6). The solid curve in figure 1 illustrates the tendency computed in m. sec.$^{-1}$ day^{-1}. The following values of the parameter were used:

1. $W=1/3\times 10^7$m., $2W\simeq 60°$ of latitude

2. $\kappa A=v_{\max}=20$ m. sec.$^{-1}$

3. $\alpha=1/(4W)$, corresponding to a slope of $\pm$ ½ of the trough and ridge lines for $y=\mp W$

It is seen that the changes predicted by the non-divergent, one-parameter model will give an increase of the wind in the middle latitudes and a decrease to the north and south. This distribution corresponds very much to the errors in the non-divergent forecasts.

We are next going to compute the changes in the mean zonal wind for the divergent, one-parameter model. Referring back to (2.8) it is seen that they may be computed from the expression:

$$\frac{\partial \overline{u}}{\partial t}=-\frac{\partial \overline{u'v'}}{\partial y}+f_0\overline{v} \tag{3.15}$$

As the first term on the right side of (3.15) is the same as the term already computed in (3.14) we may compute the term $f_0\overline{v}$ which will be a measure of the difference between the non-divergent and the divergent one-parameter models. In order to compute $\overline{v}$ we make use of the continuity equation averaged along latitude circles. This equation is

$$\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{\omega}}{\partial p}=0 \tag{3.16}$$

which in integrated form becomes

$$\bar{v}=-\int_{-W}^{y}\frac{\partial \bar{\omega}}{\partial p}dy \tag{3.17}$$

where we have used the boundary condition $\bar{v}=0$ for $y=-W$.

An expression for $\partial\bar{\omega}/\partial p$ may be obtained from the adiabatic equation, which for this simple model takes the form (Wiin-Nielsen [11]):

$$\frac{dF(p)}{dp}\frac{\partial \psi}{\partial t}+\frac{\sigma}{f_0}\omega=0. \tag{3.18}$$

Differentiation of (3.18) with respect to pressure, assuming dF/dp = constant, gives the following expression for the averaged divergence

$$\overline{\nabla\cdot\mathbf{V}}=-\frac{\partial\bar{\omega}}{\partial p}=-\frac{f_0}{\sigma^2}\frac{d\sigma}{dp}\frac{dF}{dp}\frac{\partial\bar{\psi}}{\partial t}. \tag{3.19}$$

Inserting (3.19) in (3.17) gives:

$$f_0\bar{v}=-r^2\int_{-W}^{y}\frac{\partial\bar{\psi}}{\partial t}dy;\; r^2=\frac{f_0^2}{\sigma^2}\frac{d\sigma}{dp}\frac{dF}{dp}>0. \tag{3.20}$$

We are now able to compute the influence of the mean meridional circulation on the tendency of the mean zonal wind. We have to use the expression (3.13) for the tendency of the zonally-averaged stream function, insert in (3.20), and perform the integration. The result of this procedure may be written:

$$f_o\bar{v}=-r^2\left[\frac{1}{2\mu}\left(\frac{B_2}{2\mu}-B_1\right)(1+\cos 2\mu y)+\frac{B_2}{2\mu}y\sin 2\mu y+\frac{2\mu B_1+B_2}{r^2\cosh(rW)}\cdot(\cosh(ry)-\cosh(rW))\right] \tag{3.21}$$

By a substitution of $y=+W$ it is seen that the other boundary condition $\bar{v}=0$ for $y=W$ is automatically satisfied.

The contribution of the implied mean meridional circulation to the change in the mean zonal wind is given in figure 1 as the dashed curve. The parameters used for computation of this curve are the same as before. For the parameter r^2 a value of 1.5×10^{-12} m.$^{-2}$ was used. The value is the same as estimated earlier by the author [11] or about twice the value used presently in the operational forecasts.

The curves in figure 1 show that the introduction of divergence in the one-parameter model has a tendency to counteract the effects of the meridional convergence of the transport of zonal momentum. As the transport

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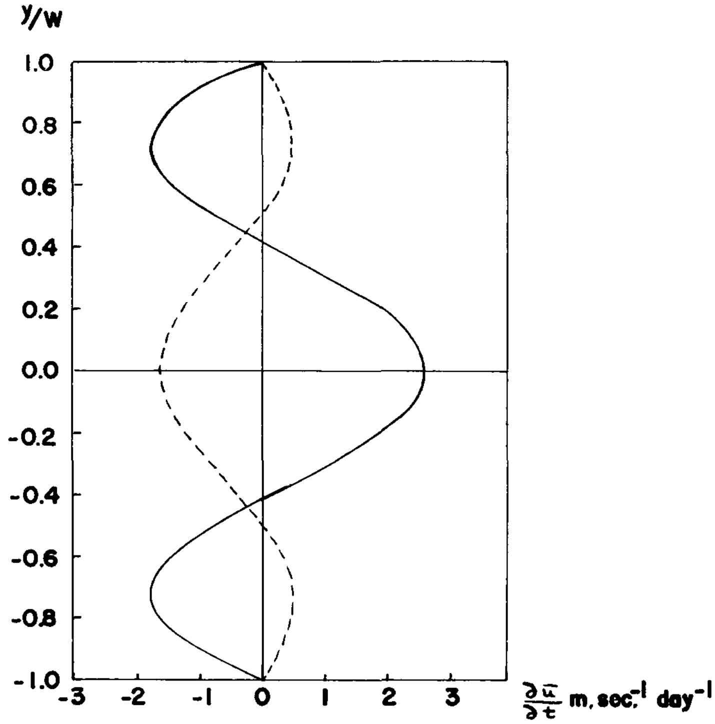

FIGURE 1.—Change in averaged zonal winds in one-parameter, divergent model in units of m. sec.$^{-1}$ day^{-1}. Solid curve is change due to convergence of meridional transport of momentum, dashed curve is change due to mean meridional circulations.

of momentum, which is the only factor in a non-divergent model, gives errors with a distribution very similar to the change caused by the momentum transport, we can expect that the errors in the mean zonal winds will be reduced in a divergent, one-parameter model.

If the idealized flow pattern treated in this section were characteristic of the atmospheric flow pattern we could expect a reduction of the error amounting to 1–2 m. sec.$^{-1}$ day^{-1}, which is roughly the order of magnitude of the present errors.

An extended series of one-parameter forecasts made with and without the contribution from the divergence does not exist. A comparison of the monthly mean errors of the averaged zonal wind for months when the divergent model has been used, with the corresponding months a year earlier when the non-divergent model was in operation, shows some reduction of the errors, but not of a magnitude comparable to that computed in our examples. The reason for this may be that the example chosen here is too extreme, especially with respect to the slope of the trough and ridge lines. It should, however, be remembered that the operational value of r^2 is about half the value chosen in the present idealized computation. Further, it is worthwhile to mention, that if an error of the present type exists in a forecast the error is likely to increase in time due to the feedback mechanism between the mean zonal current and the perturbations (see discussion in the introduction).

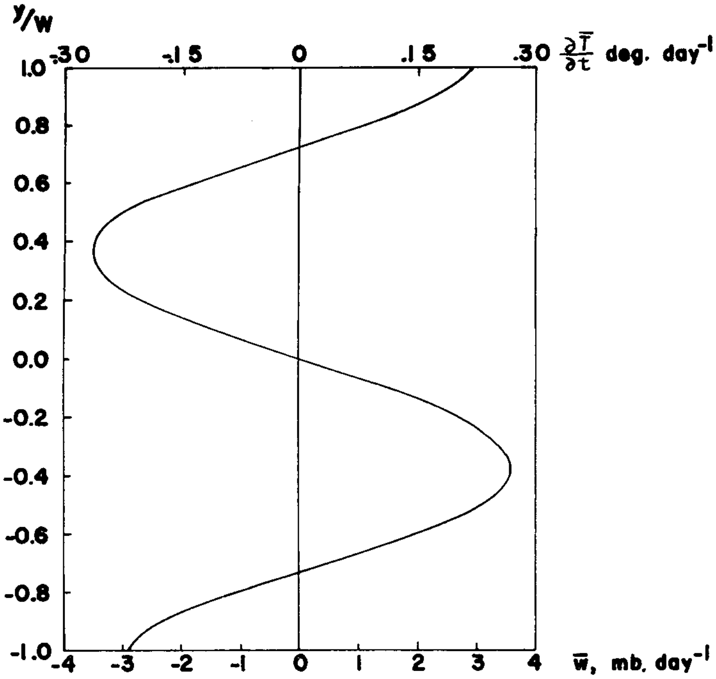

FIGURE 2.—Vertical velocity, $\omega = dp/dt$, averaged along latitude circles in one-parameter divergent model. Units: mb. day^{-1} (lower scale). Changes in zonally-averaged temperatures in the same model. Units: deg. day^{-1} (upper scale).

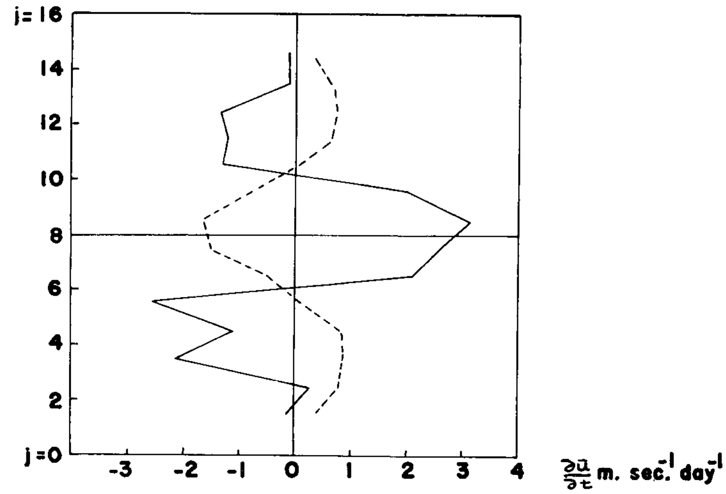

FIGURE 3.—Changes in averaged zonal winds at 500 mb. in Phillips' general circulation study. Solid curve is change due to convergence of meridional transport of momentum, dashed curve is change due to surface friction. Units: m. sec.$^{-1}$ day^{-1}.

It is interesting to compute the implied vertical velocity averaged along the latitude circles. This can be done from equation (3.18) in an averaged form. We find

$$\bar{\omega} = -\frac{f_0}{\sigma}\frac{dF}{dp}\frac{\partial \bar{\psi}}{\partial t}. \tag{3.22}$$

$\bar{\omega}$ is given as a function of "latitude" in figure 2 using numerical values of the parameters as before. The value of the coefficient $(-f_0/\sigma)(dF/dp)$ becomes 4×10^{-7}. The units for $\bar{\omega}$ are mb. day^{-1}. It is interesting to note that the distribution of $\bar{\omega}$ gives the classical picture of three meridional cells with two direct cells to the north and south and an indirect cell in the middle latitudes. The intensity of the cells may be measured by the maximum vertical velocity, which is about 3.5 mb. day^{-1} or roughly 0.5 mm. sec.$^{-1}$.

In an investigation by Phillips [7] of a two-parameter, quasi-geostrophic model it was shown that essentially the same mean vertical velocity was implied. Phillips investigated a flow pattern with no tilt and showed that the implied mean vertical motion was due to the baroclinic unstable waves, where the isotherms are lagging behind the contours. The flow pattern considered here is such that the isotherms are always parallel to the contours at the level considered. The implied mean vertical velocity is here due to the tilt of the systems and the requirement that the temperature advection should vanish. It seems therefore that the divergent, one-parameter model contains quite realistic features in this respect.

The use which has been made of the adiabatic equation in the construction of this model implies certain temperature changes at the level where the model is used. The zonally-averaged temperature changes are due only to the mean vertical velocity, because the horizontal advection of temperature is disregarded at the level. We may easily compute these temperature changes from the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right) + \sigma\bar{\omega} = 0 \tag{3.23}$$

which through use of the hydrostatic relationship reduces to

$$\frac{\partial \bar{T}}{\partial t} = \left(\frac{\sigma p}{R}\right)\bar{\omega}. \tag{3.24}$$

We find of course that the zonally averaged temperature change is proportional to the vertical velocity. The curve in figure 2 may therefore also be considered as illustrating the change in zonally averaged temperature. The value of the proportionality factor is $\sigma p/R = 0.73$ at 500 mb. The maximum change in zonally-averaged temperatures is therefore about 0.26 deg. day^{-1} for the values used to characterize the flow pattern.

We shall next compare the results obtained here for the simple divergent, one-parameter model with the result obtained by a model which contains the effects of surface friction and diabatic heating. Phillips [8] has in his general circulation experiment obtained values for the important terms in the zonal momentum budget for the two levels (750 and 250 mb.) and for the terms appearing in the thermodynamic energy equation. We shall compare our figure 1, which gives the change in zonal momentum per day due to convergence of meridional transport of zonal momentum and the meridional circulation, with figure 3,

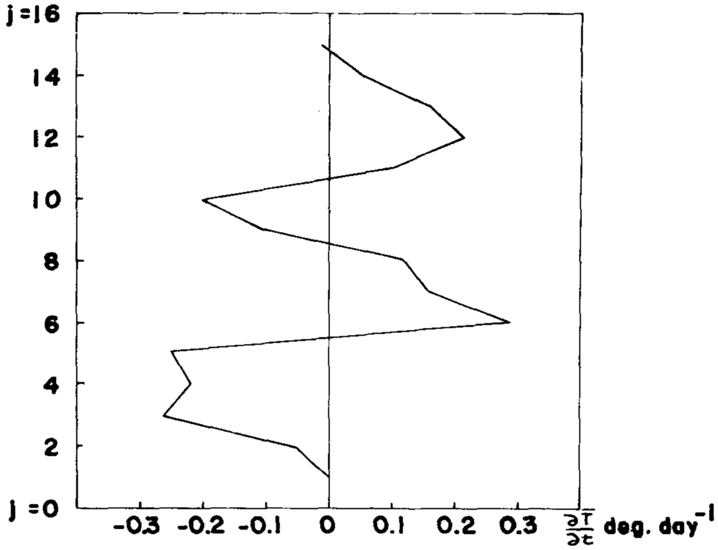

FIGURE 4.—Changes in zonally-averaged temperature at 500 mb. in Phillips' general circulation study. Units: deg. day^{-1}.

which contains the change of zonal momentum at 500 mb. in Phillips' model. The momentum budget for the 500-mb. level in his model was obtained by averaging the 250- and 750-mb. budgets, using the simple equation that

$$\frac{\partial \bar{u}_2}{\partial t}=\frac{1}{2}\left(\frac{\partial \bar{u}_1}{\partial t}+\frac{\partial \bar{u}_3}{\partial t}\right) \tag{3.25}$$

where the subscripts 1, 2, and 3 refer to 250, 500, and 750 mb. The equation becomes:

$$\frac{\partial \bar{u}_2}{\partial t}=-\frac{1}{2}\left[\frac{\partial}{\partial y}\left(\overline{u_1'v_1'}+\overline{u_3'v_3'}\right)+k\bar{u}_4\right]. \tag{3.26}$$

k is a frictional coefficient for the skin friction and u_4 the zonal wind at 1000 mb. In (3.26) we have neglected the small effect of lateral eddy viscosity used by Phillips.

The change in zonal momentum in Phillips' model at 500 mb. is due to momentum transport effects and to skin friction. The two effects are given in figure 3. Comparing figures 1 and 3 we find that the effect of meridional transport of zonal momentum acts in the same way in the two cases causing a wind increase in the middle latitudes and a decrease to the north and the south. In Phillips' model there is no effect at 500 mb. of the mean meridional circulation because of his model approximations. The effect of surface friction acts, however, opposite to the meridional transport of momentum in a way very similar to the mean meridional circulation in the model considered here. Although the divergent, one-parameter model does not have the effect of surface friction, it has an effect, which with respect to changes in mean zonal momentum acts very similar to it. Physically the two effects are very different. At every instant the frictional effect depends only on the mean zonal wind profile at the surface of the earth, while the net effect of the divergence depends upon the disturbances.

Our next problem is to compare the zonally-averaged temperature changes predicted by the divergent, one-parameter model and by the general circulation experiment. In the latter experiment the change in the mean temperature field is influenced by essentially three factors: the diabatic heating giving a cooling to the north and a warming to the south, the convergence of the meridional transport of heat, and the effect of the mean meridional circulation, disregarding again the small effect of small-scale eddy diffusion. In the model considered here we have only the effect of the mean meridional circulation. The temperature change caused by this factor is given in figure 2 (upper scale). Figure 4 gives the net effect of the three factors mentioned above. We find again that the curves in figures 2 and 4 are similar in the essential features, giving a cooling in the very low latitudes, a heating in the very high latitudes, and a strengthening of the temperature gradient in the middle latitudes. Again we are therefore tempted to conclude that the effects which are disregarded in the divergent, one-parameter model tend to balance each other with respect to changes in the zonally-averaged temperature changes.

4. TENDENCY COMPUTATIONS OF ZONAL MOMENTUM IN TWO-PARAMETER MODELS

It is generally hoped that the introduction of two or more information levels in the vertical direction will improve the short-range prediction. In this section we investigate whether we can obtain improvements in a two-parameter model in predictions of the zonal wind profile. A long record of hemispheric, two-parameter forecasts unfortunately does not exist. We therefore have to be satisfied by tendency computations.

Eliassen [5] has stressed that if the vertical structure of the atmosphere is represented by only two parameters, it becomes important to choose the two information levels with care. Eliassen represents the vertical variation of the horizontal wind by the expression

$$\mathbf{V}(x,y,p,t)=\overline{\mathbf{V}}+A(p)\mathbf{V}_T \tag{4.1}$$

where

$$\overline{\mathbf{V}}=\frac{1}{p_0}\int_0^{p_0}\mathbf{V}dp,\quad \mathbf{V}_T=\frac{1}{p_0}\int_0^{p_0}A(p)\mathbf{V}dp. \tag{4.2}$$

The two prognostic equations apply therefore to the vertically integrated flow and to the mean thermal flow in the atmosphere. If the prognostic equations are applied to specific levels in the atmosphere, Eliassen recommends choosing the two levels where $A(p)=\pm 1$ as information levels. The level where $A(p)=0$ becomes the nondivergent level. A computation leading to a determination of the function $A(p)$, based on the wind data published by Buch [2] gives the result that $A(p)=+1$ for $p=333$ mb., $A(p)=-1$ for $p=825$ mb., and $A(p)=0$ for $p=600$ mb. in winter, while the corresponding figures for his

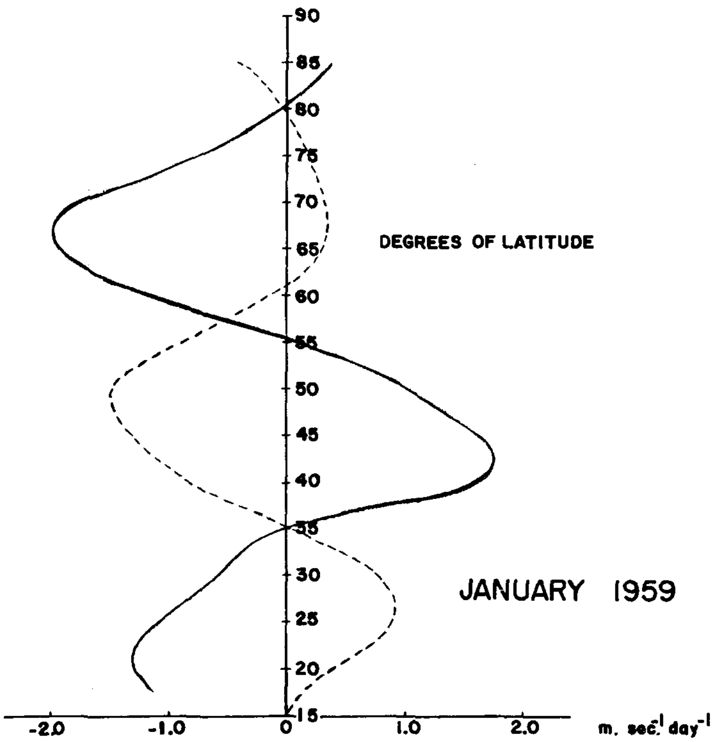

FIGURE 5.—Solid curve: 24-hour errors in mean zonal wind in operational model. Units: m. sec.$^{-1}$ day^{-1}. Dashed curve: Contributions from mean meridional circulation to changes in mean zonal winds, averaged for the month of January 1959. Units: m. sec.$^{-1}$ day^{-1}. Vertical coordinate is latitude.

summer data become: $A(p)=+1$ for $p=367$ mb., $A(p)=-1$ for $p=833$ mb., and $A(p)=0$ for $p=567$ mb. As we here intend to make a tendency computation for zonal momentum in winter, we can assume that the 600-mb. level is the non-divergent level. For our computation we will need values of the vertical motion at this level. Thompson [9] has earlier arrived at the conclusion that the non-divergent level can be approximated by the 600-mb. level. The initial vertical velocities presently computed on a daily basis apply therefore at that level.

We are next going to see how we can make a tendency computation for the zonal momentum for the 500-mb. level based upon a two-parameter model. Referring back to equation (2.3) we obtain in the frictionless case:

$$\frac{\partial\bar{u}}{\partial t}=-\frac{\partial\overline{uv}}{\partial y}+f\bar{v}-\frac{\partial\overline{u\omega}}{\partial p}. \tag{4.3}$$

We are here interested in an estimate of the contributions from the mean meridional circulation and the vertical advection of momentum to the rate of change of zonal momentum; i.e., the magnitude of the second and third terms on the right side of (4.3).

The contribution from the mean meridional circulation may be computed by a method similar to the one applied in section 3. From the continuity equation averaged along latitude circles we obtain by integration:

$$\bar{v}_5=\bar{v}_{5,b}-\int_0^y\left(\overline{\frac{\partial\omega}{\partial p}}\right)_5 dy \tag{4.4}$$

where $\bar{v}_5$ is the mean meridional wind component at 500 mb., $\bar{v}_{5,b}$ the same component at the southern boundary, and $-(\overline{\partial\omega/\partial p})_5$ the averaged component of the divergence at the same level. For the vertical derivative of the vertical velocity we have in accordance with the model approximations in the JNWP operational model:

$$\left(\frac{\partial\omega}{\partial p}\right)_5=\frac{\omega_6}{2P} \tag{4.5}$$

where $P=40$ cb. (4.5) is exact if the vertical velocity has a parabolic distribution with zero-points at 1000 and 200 mb. and an extremum value at 600 mb. It should be stressed that we get the same result with a very good approximation if we derive the vertical distribution of horizontal divergence with Eliassen's method [5]. He assumes that this distribution can be obtained from equation (4.1) simply by applying the operator $\nabla\cdot$ to both sides; i.e.

$$\nabla\cdot\mathbf{V}=A(p)\nabla\cdot\mathbf{V}_T. \tag{4.6}$$

Using the function $A(p)$ derived from Buch's [2] winter data we get $A(p_5)=0.4$.

The component $\bar{v}_{5,b}$ at the southern boundary is very difficult if not impossible to determine from the available data. In the computations to be described below it has been assumed that $\bar{v}_{5,b}=0$, which corresponds to a rigid wall at the boundary. This boundary condition does not correspond to the one in the real atmosphere nor to the one applied in the computations of the vertical velocity. A value of $\bar{v}_{5,b}$ different from zero would, however, only shift the reference point of the curves to be described later. The mean meridional velocity was therefore computed from the formula

$$\bar{v}_5=-\frac{1}{2P}\int_0^y\bar{\omega}_6 dy. \tag{4.7}$$

The vertical velocity, computed operationally, was averaged in latitude rings with a width of 5° latitude for each day of the month January 1959, as described by the author [12] in an earlier paper. $\bar{v}_5$ was then computed by a numerical integration of the expression (4.7). The results are presented in figure 5, where the contribution from the mean meridional circulation to the change in the mean zonal current is plotted as the change which would occur in $\bar{u}$ in 24 hours provided the initial mean meridional circulation persisted for this time; i.e.

$$\Delta\bar{u}_5=\Delta t\cdot f_0\cdot\bar{v}_5 \tag{4.8}$$

where $\Delta t=24$ hours.

The monthly errors in the average zonal winds for the 24-hour forecasts have been entered in the diagram on figure 5. It is obvious from figure 5 that the mean meridional circulation as computed from the procedure described above would decrease the errors in the average zonal winds in most latitudes. It accounts, at least in the monthly average presented here, for the greater part of the errors in the middle and low latitudes, while the contribution from the meridional circulation to the correction of the errors in the high latitudes is much too small, although of the correct sign. It seems therefore that a two-parameter model with the proper selection of the information levels, and especially the level where the divergence is assumed to be zero, will in the average reduce the errors found in non-divergent one-parameter forecasts. The rather small correction to the errors in the averaged zonal winds that is found in the high latitudes points in the direction that the two-parameter model predicts too small mean meridional circulations in these latitudes. This may also be expressed by saying that the level of non-divergence is lower in these high latitudes than it is farther to the south. If this is the case one will need more than two parameters to correct for the total error in the zonally-averaged winds.

The result that the level of non-divergence is lower in the high latitudes is opposite to the results obtained by Landers [6], but agrees in general with the climatological variation of the tropopause with latitude. It is reasonable to expect that the first mode of the vertical velocity in the troposphere, which is the only one used here, to a large extent is determined by the tropopause. The great similarity which is found between the mean meridional circulation derived from vertical velocities computed from a two-parameter model and the mean meridional circulation implied by the divergent one-parameter model investigated in section 3 for flow patterns with opposite tilt of trough and ridge lines in the high and low latitudes must mean that the divergent one-parameter model has a pattern of vertical velocity and divergence quite similar to that derived from a two-parameter model, at least when averaged along latitude circles.

In view of this result, suggested by the above investigation of mean meridional circulation in the two models, it is interesting and instructive to compare vertical velocities in the two models. Let us for this purpose restrict ourselves to simple harmonic waves. The vertical velocity in the divergent one-parameter model may be found from two equations:

$$\frac{\partial \nabla^2 \psi_1}{\partial t} + \mathbf{V}_1 \cdot \nabla(\nabla^2 \psi_1 + f) = q_1 \frac{\partial \psi_1}{\partial t} \tag{4.9}$$

$$\omega_1 = r_1 \frac{\partial \psi_1}{\partial t}. \tag{4.10}$$

where $q_1 = \frac{f_0^2}{\sigma^2} \frac{d\sigma}{dp} \frac{dF}{dp}$, $r_1 = \frac{f_0}{\sigma}\left(-\frac{dF}{dp}\right)$. In these formulae ψ_1 is the stream function, $\mathbf{V}_1 = \mathbf{k} \times \nabla \psi_1$ the horizontal wind, f_0 is a standard value of the Coriolis parameter, $\sigma = -\alpha \partial \ln \theta / \partial p$, and $F(p)$ is a function describing the variation of the horizontal wind with pressure. For the details in the derivation of equations (4.9) and (4.10) the reader is referred to an earlier paper by the author [11]. The numerical values to be used in the following are:

$$q_1 = 1.5 \times 10^{-12} \text{m.}^{-2}$$

$$r_1 = \frac{3}{8} \times 10^{-6} \text{ tm.}^{-3} \text{ sec.}^{-1}$$

Assume now the simple sinusoidal flow pattern to be defined by:

$$\psi_1(x,y,t) = -U_1 y + A \sin \kappa (x - ct). \tag{4.11}$$

We find from (4.9) the wave-speed formula:

$$c_1 = \frac{U_1 - \beta/\kappa^2}{1 + q_1/\kappa^2}. \tag{4.12}$$

From (4.10) we find

$$\omega_1 = -r_1 \cdot c_1 \cdot v_1, \quad v_1 = \frac{\partial \psi_1}{\partial x} \tag{4.13}$$

which combined with (4.12) leads to:

$$\omega_1 = -r_1 \cdot \frac{U_1 - \beta/\kappa^2}{1 + q_1/\kappa^2} v_1. \tag{4.14}$$

Equation (4.14) shows that we have an implied vertical motion which is upward ($\omega < 0$) between trough and ridge and downward ($\omega > 0$) between ridge and trough for waves with a wavelength so short that $U_1 - \beta/\kappa^2$ is positive, while the opposite is true for waves with a wavelength larger than the stationary wavelength. Inserting typical values of U and v it is seen, that the implied vertical motion can obtain values of a few cm. sec.$^{-1}$. The absolute values decrease from the very short waves to the stationary wave and increase then again as the wavelength becomes larger. For comparison with a later result we shall write (4.14) in the form:

$$\omega_1 = -\frac{r_1 U_1}{1 + q_1/\kappa^2} v_1 + \frac{r_1}{1 + q_1/\kappa^2} \cdot \frac{\beta}{\kappa^2} v_1. \tag{4.15}$$

This vertical velocity can be compared with the implied vertical velocity in the two-parameter model. It has been shown earlier by the author [10] that this vertical velocity is given by the expression:

$$\omega_2 = -\frac{r_2(2U')}{1 + q_2/\kappa^2} v_2 + \frac{r_2}{1 + q_2/\kappa^2} \cdot \frac{\beta}{\kappa^2} \cdot v' \tag{4.16}$$

for simple sinusoidal waves. The symbols in (4.16) have the following meaning: v_2 and v' are the meridional wind components for the mean flow and the thermal flow in the two-parameter model, while $r_2 = 2f_0/\sigma P$, $q_2 = 2f_0^2/\sigma P^2$. Comparing (4.15) with (4.16) it is seen that the first terms in the two expressions are very similar. They both give a contribution to the vertical velocity which is in phase

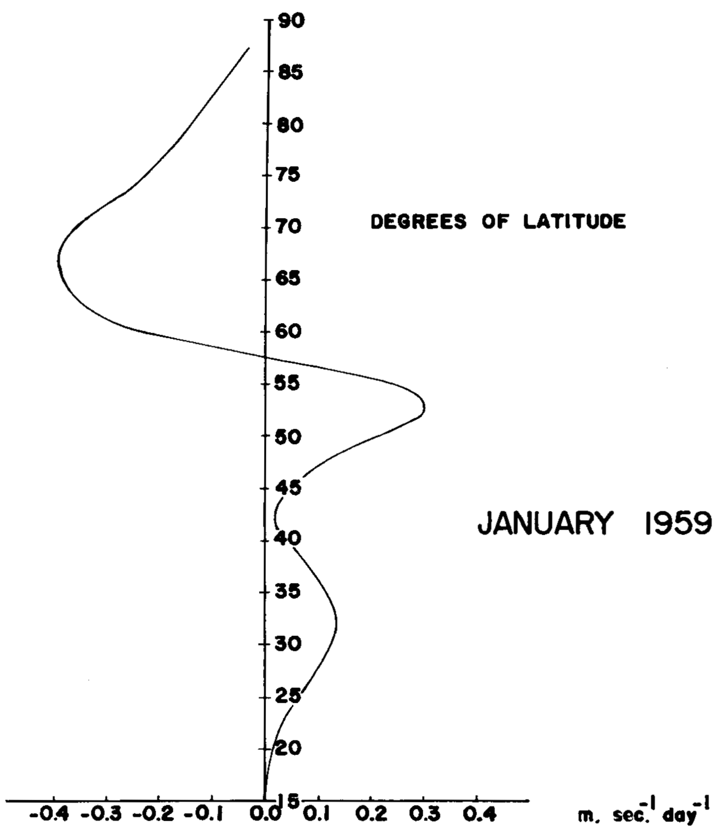

FIGURE 6.—Contribution from vertical advection of momentum to changes in the mean zonal wind as a function of latitude. Units: m. sec.$^{-1}$ day^{-1}. Averaged contribution for January 1959.

with the meridional component of the flow (v_1 and v_2). The contribution from the second term in the expression is in the one-parameter model in phase with the v_1-wave, while it is in phase with the thermal wave (v') in the two-parameter model. In the majority of cases, where the v_1-wave and v'-wave have only small phase differences, it is seen the divergent one-parameter model will imply vertical velocities that are distributed in nearly the same manner as those obtained from a two-parameter model. The magnitude of the vertical velocities will be somewhat, but not very much, different in the two models, because $r_1<r_2$, $q_1<q_2$ using values from the standard atmosphere.

We have thus in this section shown that the contribution from the mean meridional circulation (in a two-parameter model with the non-divergent level at 600 mb.) to changes in zonal momentum at 500 mb. will reduce the errors in the prediction of the averaged zonal winds in the non-divergent one-parameter model. The same conclusion was made plausible in section 3, where the divergent one-parameter model was investigated. It is *not* claimed that the contribution from the meridional circulation can explain the total error found in the prediction of the averaged zonal flow, but it has been the purpose to show that the two models which have been investigated give similar reductions of the errors. The contribution from the surface friction is naturally of importance, and it is evident that this contribution can be incorporated with greater accuracy in a two- or multi-parameter model.

5. ESTIMATES OF CONTRIBUTIONS FROM VERTICAL ADVECTION

The effects of the vertical advection term in the equations of motion, which corresponds to the vertical advection of vorticity and the twisting term in the vorticity equation, have in general not been included in the forecasts. We may get an estimate of the importance of these terms in a two-parameter model for the prediction of averaged zonal winds by a computation of the last term in equation (4.3), which can be estimated using the vertical velocities computed operationally. It may be argued that the vertical velocities are computed from a model not containing these terms. It should be pointed out that although this argument is true with regard to predictions, it does not apply initially, because the contribution from the vertical advection of vorticity and the twisting cancel out in the thermal vorticity equation in a two-parameter model. Initially, we will therefore have the same vertical velocities. The term in question may be approximated in a way similar to the one applied in section 4. We get:

$$\left(\frac{\partial\overline{\omega u}}{\partial p}\right)_5=-\left(\overline{u_5\frac{\omega_6}{2P}}-\overline{\omega_5\cdot\frac{u_T}{\Delta p}}\right) \tag{5.1}$$

where $P=40$ cb., u_T is the zonal component of the thermal wind between 850 and 500 mb., $\Delta p=35$ cb., and the subscripts denote the level where the quantities are measured. The contribution from both of these terms to the change in the zonal wind can easily be computed from the available data. At each grid point we have to form the product of the vertical velocity and the zonal component of the 500-mb. wind and the thermal wind, respectively. These products are then averaged in the latitude rings and the results expressed as changes in the averaged zonal winds in a 24-hour period. The result of these computations is shown in figure 6 for the month of January 1959. The magnitude of the contribution from the vertical advection terms is first of all smaller than that from the convergence of the meridional transport of momentum and the mean meridional circulation. Secondly, we find that the vertical advection of momentum is not distributed in such a way as to reduce the errors in the non-divergent, one-parameter model. The obvious conclusion from this computation is that the contribution from the vertical advection of momentum will not improve the forecast of zonally averaged winds on the average in a two-parameter model. The effect is definitely of smaller magnitude than the other effects in equation (4.3), and it is likely that a better resolution in the vertical is needed to incorporate this effect with greater accuracy.

The term containing the vertical transport of momentum can therefore possibly be neglected in the midtroposphere as indicated above. However, approaching the ground the effect must become of greater importance simply because the vertical transport of momentum next to the ground is the surface stress. Considerable information regarding the importance of the surface stress for the momentum budget in the lower layers has been computed by Phillips [8].

6. GENERAL CONCLUSIONS

The ability of one- and two-parameter models to predict changes in averaged zonal winds has been investigated. It has been found that the divergent, one-parameter model will reduce the errors found in non-divergent, one-parameter predictions in the average due to the implied mean meridional circulations found in flow patterns where the trough and ridge lines have opposite tilts in the low and high latitudes.

Tendency computations of changes in zonally averaged winds have been performed using initial values of vertical velocities computed from a frictionless, two-parameter model with no divergence at 600 mb. It is found that the mean meridional circulation in this model also would reduce the errors found in the non-divergent, one-parameter predictions in the average. One result of this part of the study is that it is very important to choose the information levels properly, if a two-parameter model is used. As the contribution from the surface friction also seems to be important for the budget of zonal momentum even in short-range prediction, and because the surface flow is difficult to represent with good accuracy in a two-parameter model, it is likely that more than two parameters will be necessary to predict changes in zonal momentum correctly.

ACKNOWLEDGMENTS

The author would like to thank Mrs. Margaret McLaughlin for the coding of the computations in sections 4 and 5.

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